
CHAI'S CONJECTURE AND FUBINI PROPERTIES OF DIMENSIONAL MOTIVIC INTEGRATION

by

Raf Cluckers, François Loeser & Johannes Nicaise

Abstract. — We prove that a conjecture of Chai on the additivity of the base change conductor for semi-abelian varieties over a discretely valued field is equivalent to a Fubini property for the dimensions of certain motivic integrals. We prove this Fubini property when the valued field has characteristic zero.

1. Introduction

Let R be a henselian discrete valuation ring with quotient field K and perfect residue field k . Let G be a semi-abelian variety over K , i.e., an extension of an abelian K -variety by a K -torus. Then G can be canonically extended to a smooth separated commutative group scheme \mathcal{G} over R , the so-called Néron *lft*-model of G [4, 10.1.1]. We say that G has semi-abelian reduction if the identity component of the special fiber of \mathcal{G} is a semi-abelian k -variety.

In [7], Chai introduced the *base change conductor* $c(G)$ of G , a positive rational number that measures the defect of semi-abelian reduction of G . Its precise definition is recalled in Definition 2.3.1. The base change conductor vanishes if and only if G has semi-abelian reduction. For algebraic tori, this invariant had previously been defined and studied by Chai and Yu [8]. They proved the deep result that the base change conductor of a K -torus T is invariant under isogeny. Applying an argument from [12], they deduced that $c(T)$ equals one half of the Artin conductor of the cocharacter module of T . For semi-abelian varieties, however, no similar cohomological interpretation is known to hold in general; in fact, the base change conductor is not even invariant under isogeny [7, §6.10], and many of its properties remain mysterious. One of the main open questions is the following conjecture, formulated by Chai in [7, §8.1].

1.1. Conjecture (Chai). — *Let G be a semi-abelian K -variety which fits into an exact sequence of algebraic K -groups*

$$0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0$$

with T a K -torus and A an abelian K -variety. Then we have

$$c(G) = c(A) + c(T).$$

The fundamental difficulty underlying this conjecture is that an exact sequence of semi-abelian varieties does not give rise to an exact sequence of Néron *lft*-models, in general. Chai proved the conjecture if k is finite, using Fubini's theorem for integrals with respect to the Haar measure on the completion of K . He also proved the conjecture when K has mixed characteristic, using a different method and applying the property that $c(T)$ only depends on the isogeny class of T . If k has characteristic zero (more generally, if G obtains semi-abelian reduction after a *tame* finite extension of K), Chai's conjecture can be proven in an elementary way; see [15, 4.23].

In the first part of the present paper, we show that, in arbitrary characteristic, Chai's conjecture is equivalent to a Fubini property for the dimensions of certain *motivic* integrals (equation (4.2.1) in Theorem 4.2.1). We then prove in the second part of the paper that this Fubini property holds when K has characteristic zero (Theorem 4.2.3). This yields a new proof of the conjecture in that case, which is close in spirit to Chai's proof of the finite residue field case.

The strength of our approach lies in the fact that we combine insights of two theories of motivic integration, namely, the geometric theory of motivic integration on rigid varieties of Loeser and Sebag [16] and the model-theoretic approach of Cluckers and Loeser [9, 10]. Let us emphasize that the Fubini property in (4.2.1) is not an immediate corollary of the Fubini results from [10]; see Remark 4.2.5. We need to combine the theory in [10] with a new result (Theorem 5.2.1 and its corollary), which roughly states that the virtual dimension of a motivic integral over a fixed space only depends on the dimensions of the values of the integrand. This theorem may be of independent interest.

We hope that our reformulation of Chai's conjecture in terms of motivic integrals will also shed new light on the open case of the conjecture, when k is infinite, K has positive characteristic and G is wildly ramified.

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2. Preliminaries

2.1. Notation. — Throughout this article, R denotes a Henselian discrete valuation ring with quotient field K and perfect residue field k . We denote by \mathfrak{m} the maximal ideal of R , by R^{sh} a strict henselization of R and by K^{sh} its field of fractions. The residue field k^s of R^{sh} is an algebraic closure of k . We denote by \widehat{R} the \mathfrak{m} -adic completion of R and by \widehat{K} its field of fractions.

For every ring A , we denote by (Sch/A) the category of A -schemes. We consider the *special fiber functor*

$$(\cdot)_k : (\text{Sch}/R) \rightarrow (\text{Sch}/k) : \mathcal{X} \mapsto \mathcal{X}_k = X \times_R k$$

and the *generic fiber functor*

$$(\cdot)_K : (\text{Sch}/R) \rightarrow (\text{Sch}/K) : \mathcal{X} \mapsto \mathcal{X}_K = \mathcal{X} \times_R K.$$

A variety over a ring A is a reduced separated A -scheme of finite type.

2.2. Néron models and semi-abelian reduction. — A semi-abelian variety over a field F is an extension of an abelian F -variety by an algebraic F -torus. Let G be a semi-abelian variety over K . It follows from [4, 10.2.2] that G admits a Néron *lft*-model \mathcal{G} in the sense of [4, 10.1.1]. It is the minimal extension of G to a smooth separated group scheme over R . We say that G has semi-abelian reduction if the identity component \mathcal{G}_k^o of the special fiber of \mathcal{G} is a semi-abelian k -variety. There always exists a finite separable extension L of K such that $G \times_K L$ has semi-abelian reduction. If G is an abelian variety, then this is Grothendieck's Semi-Stable Reduction Theorem [3, IX.3.6]. If G is a torus, then one can take for L the splitting field of G . The general case is easily deduced from these special cases; see [14, 3.11].

Let K' be a finite separable extension of K , and denote by R' the integral closure of R in K' . We set $G' = G \times_K K'$ and we denote by \mathcal{G}' the Néron *lft*-model of G' . By the universal property of the Néron *lft*-model, there exists a unique morphism of R' -schemes

$$(2.2.1) \quad h : \mathcal{G} \times_R R' \rightarrow \mathcal{G}'$$

that extends the natural isomorphism between the generic fibers. If G has semi-abelian reduction, then h is an open immersion [3, 3.1(e)], which induces an isomorphism

$$(\mathcal{G} \times_R R')^o \rightarrow (\mathcal{G}')^o$$

between the identity components of $\mathcal{G} \times_R R'$ and \mathcal{G}' [1, VI_B.3.11].

2.3. The base change conductor. — Let G be a semi-abelian variety over K . Let K' be a finite separable extension of K such that $G' = G \times_K K'$ has semi-abelian reduction, and denote by $e(K'/K)$ the ramification index of K' over K . The morphism (2.2.1) induces an injective morphism

$$(2.3.1) \quad \text{Lie}(h) : \text{Lie}(\mathcal{G}) \otimes_R R' \rightarrow \text{Lie}(\mathcal{G}')$$

of free R' -modules of rank $\dim(G)$.

2.3.1. Definition (Chai [7], Section 1). — The base change conductor of G is defined by

$$c(G) = \frac{1}{e(K'/K)} \cdot \text{length}_{R'}(\text{coker}(\text{Lie}(h))).$$

This definition does not depend on the choice of K' . The base change conductor is a positive rational number that vanishes if and only if G has semi-abelian reduction [15, 4.16]. One can view $c(G)$ as a measure for the defect of semi-abelian reduction of G .

2.4. A generalization of Chai's conjecture. — In [7, 8.1], Chai asks whether Conjecture 1.1 can be generalized as follows.

2.4.1. Question. — Do we have $c(G_2) = c(G_1) + c(G_3)$ for every exact sequence of semi-abelian K -varieties

$$0 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow 0?$$

If G_1 , G_2 and G_3 are tori, this can be easily deduced from the deep fact that the base change conductor of a torus is one half of the Artin conductor of the cocharacter module [8], in the following way.

2.4.2. Proposition. — *Let*

$$0 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow 0$$

be an exact sequence of K -tori. Then $c(G_2) = c(G_1) + c(G_3)$.

Proof. — The sequence of cocharacter modules

$$0 \rightarrow X_\bullet(G_1) \rightarrow X_\bullet(G_2) \rightarrow X_\bullet(G_3) \rightarrow 0$$

is exact. Tensoring with \mathbb{Q} , we get a split exact sequence of $\mathbb{Q}[\text{Gal}(L/K)]$ -modules

$$0 \rightarrow X_\bullet(G_1) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow X_\bullet(G_2) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow X_\bullet(G_3) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow 0$$

where L is the splitting field of G_2 . Thus the Artin conductor of $X_\bullet(G_2) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the sum of the Artin conductors of $X_\bullet(G_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $X_\bullet(G_3) \otimes_{\mathbb{Z}} \mathbb{Q}$. Since the base change conductor of a torus is one half of the Artin conductor of the cocharacter module [8], we find that $c(G_2) = c(G_1) + c(G_3)$. \square

2.4.3. Corollary. — *If Conjecture 1.1 holds, then Question 2.4.1 has a positive answer when G_1 is a torus.*

Proof. — Assume that G_1 is a torus. For every semi-abelian K -variety G , we denote by G_{tor} its maximal subtorus and by $G_{\text{ab}} = G/G_{\text{tor}}$ its abelian part. We consider the closed subgroup $\tilde{G}_2 = (G_3)_{\text{tor}} \times_{G_3} G_2$ of G_2 . We have a short exact sequence of K -groups

$$(2.4.1) \quad 0 \rightarrow G_1 \rightarrow \tilde{G}_2 \rightarrow (G_3)_{\text{tor}} \rightarrow 0$$

so that \tilde{G}_2 is an extension of K -tori, and thus a torus. Moreover, the morphism

$$G_2/\tilde{G}_2 \rightarrow G_3/(G_3)_{\text{tor}} = (G_3)_{\text{ab}}$$

is an isomorphism, so that $\tilde{G}_2 = (G_2)_{\text{tor}}$ and $(G_2)_{\text{ab}} \cong (G_3)_{\text{ab}}$. By Conjecture 1.1, we have $c(G_i) = c((G_i)_{\text{tor}}) + c((G_i)_{\text{ab}})$ for $i = 2, 3$. Applying Proposition 2.4.2 to

the sequence (2.4.1), we find that $c((G_2)_{\text{tor}}) = c(G_1) + c((G_3)_{\text{tor}})$. It follows that $c(G_2) = c(G_1) + c(G_3)$. \square

Below, we will follow a different approach. We will use the invariance of the base change conductor of a torus under isogeny to reduce Question 2.4.1 to the case where the maximal split subtorus $(G_3)_{\text{sp}}$ of G_3 is trivial (of course, this is always the case if G_3 is an abelian variety as in Conjecture 1.1). Then we prove that, if G_1 is a torus and $(G_3)_{\text{sp}}$ is trivial, the additivity property of the base change conductor in Question 2.4.1 is equivalent to a certain Fubini property for motivic integrals. We prove this Fubini property when K has characteristic zero. These arguments do not use the invariance of the base change conductor of a torus under isogeny.

3. Motivic Haar measures on semi-abelian varieties

3.1. The Grothendieck ring of varieties. — Let F be a field. We denote by $K_0(\text{Var}_F)$ the *Grothendieck ring of varieties over F* . As an abelian group, $K_0(\text{Var}_F)$ is defined by the following presentation:

- *generators*: isomorphism classes $[X]$ of separated F -schemes of finite type X ,
- *relations*: if X is a separated F -scheme of finite type and Y is a closed subscheme of X , then

$$[X] = [Y] + [X \setminus Y].$$

These relations are called *scissor relations*.

By the scissor relations, one has $[X] = [X_{\text{red}}]$ for every separated F -scheme of finite type X , where X_{red} denotes the maximal reduced closed subscheme of X . We endow the group $K_0(\text{Var}_F)$ with the unique ring structure such that

$$[X] \cdot [X'] = [X \times_F X']$$

for all separated F -schemes of finite type X and X' . The identity element for the multiplication is the class $[\text{Spec } F]$ of the point. To any constructible subset C of a separated F -scheme of finite type X , one can associate an element $[C]$ in $K_0(\text{Var}_F)$ by choosing a finite partition of C into subvarieties C_1, \dots, C_r of X and setting $[C] = [C_1] + \dots + [C_r]$. The scissor relations imply that this definition does not depend on the choice of the partition. For a detailed survey on the Grothendieck ring of varieties, we refer to [20].

We denote by $K_0^{\text{mod}}(\text{Var}_F)$ the *modified Grothendieck ring of varieties over F* [20, §3.8]. This is the quotient of $K_0(\text{Var}_F)$ by the ideal \mathcal{I}_F generated by elements of the form $[X] - [Y]$ where X and Y are separated F -schemes of finite type such that there exists a finite, surjective, purely inseparable F -morphism $Y \rightarrow X$. If F has characteristic zero, then it is easily seen that \mathcal{I}_F is the zero ideal [20, 3.11], so that $K_0(\text{Var}_F) = K_0^{\text{mod}}(\text{Var}_F)$. It is not known if \mathcal{I}_F is non-zero if F has positive characteristic. In particular, if F' is a non-trivial finite purely inseparable extension of F , it is not known whether $[\text{Spec } F'] \neq 1$ in $K_0(\text{Var}_F)$.

There exists a canonical isomorphism from $K_0^{\text{mod}}(\text{Var}_F)$ to the Grothendieck ring $K_0(\text{ACF}_F)$ of the theory ACF_F of algebraically closed fields over F [20, 3.13]. One may also consider the semi-ring variant $K_0^+(\text{ACF}_F)$ of the ring $K_0(\text{ACF}_F)$, defined as follows. Let $\mathcal{L}_{\text{ring}}(F)$ be the ring language with coefficients from F . As a semi-group, $K_0^+(\text{ACF}_F)$ is the quotient of the free commutative semi-group generated by a symbol $[X]$ for each $\mathcal{L}_{\text{ring}}(F)$ -definable set, with zero-element $[\emptyset]$, and divided out by the relations

- if X and Y are $\mathcal{L}_{\text{ring}}(F)$ -definable subsets of a common $\mathcal{L}_{\text{ring}}(F)$ -definable set, then

$$[X \cup Y] + [X \cap Y] = [X] + [Y];$$

- if there exists an $\mathcal{L}_{\text{ring}}(F)$ -definable bijection $X \rightarrow Y$ for the theory ACF_F , then $[X] = [Y]$.

The semi-group $K_0^+(\text{ACF}_F)$ carries a structure of semi-ring, induced by taking Cartesian products, $[X][Y] = [X \times Y]$.

If R has equal characteristic, then we put

$$K_0^R(\text{Var}_k) = K_0(\text{Var}_k).$$

If R has mixed characteristic, then we put

$$K_0^R(\text{Var}_k) = K_0^{\text{mod}}(\text{Var}_k).$$

We denote by \mathbb{L} the class of the affine line \mathbb{A}_k^1 in $K_0^R(\text{Var}_k)$ and also in $K_0^+(\text{ACF}_k)$. We will write \mathcal{M}_k^R for the localization of $K_0^R(\text{Var}_k)$ with respect to \mathbb{L} , and \mathcal{M}_k^+ for the localization of $K_0^+(\text{ACF}_k)$ with respect to \mathbb{L} and the elements $\mathbb{L}^i - 1$ for all $i > 0$.

For every element α of $K_0^R(\text{Var}_k)$, we denote by $P(\alpha)$ its *Poincaré polynomial* [20, 4.13]. This is an element of $\mathbb{Z}[T]$, and the map

$$P : K_0^R(\text{Var}_k) \rightarrow \mathbb{Z}[T] : \alpha \mapsto P(\alpha)$$

is a ring morphism. Hence, the map

$$P^+ : K_0^+(\text{ACF}_k) \rightarrow \mathbb{Z}[T],$$

obtained by composing P with the canonical morphism $K_0^+(\text{ACF}_F) \rightarrow K_0(\text{ACF}_F) \cong K_0^R(\text{Var}_k)$, is a morphism of semi-rings. When α is the class of a separated k -scheme of finite type X , then for every $i \in \mathbb{N}$, the coefficient of T^i in $P(\alpha)$ is $(-1)^i$ times the i -th *virtual Betti number* of X . The degree of $P(\alpha)$ is twice the dimension of X [18, 8.7].

We have $P(\mathbb{L}) = T^2$, so that P localizes through a ring morphism

$$P : \mathcal{M}_k^R \rightarrow \mathbb{Z}[T, T^{-1}]$$

and P^+ localizes through a semi-ring morphism

$$P^+ : \mathcal{M}_k^+ \rightarrow \mathbb{Z}[T, T^{-1}, (T^{2i} - 1)^{-1}]_{i>0}.$$

3.1.1. Definition. — Let α be an element of \mathcal{M}_k^R , resp. of \mathcal{M}_k^+ . We define the *virtual dimension* of α as $1/2$ times the degree of the Poincaré polynomial $P(\alpha)$,

resp. $P^+(\alpha)$, with the convention that the degree of the zero polynomial is $-\infty$ and $(1/2) \cdot (-\infty) = -\infty$. We denote the virtual dimension of α by $\dim(\alpha)$.

By definition, the virtual dimension is an element of $(1/2) \cdot \mathbb{Z} \cup \{-\infty\}$. For every separated k -scheme of finite type X and every integer i , we have

$$\dim([X]\mathbb{L}^i) = \dim(X) + i.$$

3.2. Motivic integration on K -varieties. — Let X be a K -variety. We say that X is *bounded* if $X(K^{sh})$ is bounded in X in the sense of [4, 1.1.2]. If X is a smooth K -variety, then by [4, 3.4.2 and 3.5.7], X is bounded if and only if X admits a *weak Néron model* \mathcal{X} . This means that \mathcal{X} is a smooth R -variety endowed with an isomorphism $\mathcal{X}_K \rightarrow X$ such that the natural map

$$\mathcal{X}(R^{sh}) \rightarrow X(K^{sh})$$

is a bijection.

The theory of motivic integration on rigid varieties was developed in [16], and further extended in [19] and [17]. We refer to [21] for a survey; see in particular [21, §2.4] for an erratum to the previous papers. One of the main results can be reformulated for algebraic varieties as follows. Let X be bounded smooth K -variety of pure dimension, and let ω be a gauge form on X , i.e., a nowhere vanishing differential form of degree $\dim(X)$. Let \mathcal{X} be a weak Néron model for X . For every connected component C of $\mathcal{X}_k = \mathcal{X} \times_R k$, we denote by $\text{ord}_C \omega$ the order of ω along C . If ϖ is a uniformizer in R , then $\text{ord}_C \omega$ is the unique integer n such that $\varpi^{-n} \omega$ extends to a generator of $\Omega_{\mathcal{X}/R}^{\dim(X)}$ at the generic point of C .

3.2.1. Theorem-Definition. — The object

$$(3.2.1) \quad \int_X |\omega| = \mathbb{L}^{-\dim(X)} \sum_{C \in \pi_0(\mathcal{X}_k)} [C] \mathbb{L}^{-\text{ord}_C \omega} \in \mathcal{M}_k^R$$

only depends on X and ω , and not on the choice of a weak Néron model \mathcal{X} . We call it the motivic integral of ω on X .

Proof. — By [18, 4.9], the formal \mathfrak{m} -adic completion of \mathcal{X} is a formal weak Néron model of the rigid analytification X^{rig} of $X \times_K \widehat{K}$, so that the result follows from [15, 2.3]. \square

It is clear from the definition that the motivic integral of ω on X remains invariant if we multiply ω with a unit in R .

3.2.2. Remark. — In the literature, the factor $\mathbb{L}^{-\dim(X)}$ in the right hand side of (3.2.1) is sometimes omitted (for instance in [21]); this depends on the choice of a normalization for the motivic measure.

3.3. Motivic Haar measures. — Consider a semi-abelian K -variety G of dimension g . We denote by \mathcal{G} the Néron *lft*-model of G and by Ω_G the free rank one R -module of translation-invariant differential forms in $\Omega_{\mathcal{G}/R}^g(\mathcal{G})$. Note that $\Omega_G \otimes_R K$ is canonically isomorphic to the K -vector space of translation-invariant differential forms of maximal degree on G , so that we can view Ω_G as an R -lattice in this vector space. We denote by ω_G a generator of Ω_G . It is unique up to multiplication with a unit in R .

Let K' be a finite separable extension of K such that $G' = G \times_K K'$ has semi-abelian reduction, and let d be the ramification index of K' over K . Denote by R' the normalization of R in K' . Dualizing the morphism (2.3.1) and taking determinants, we find a morphism of free rank one R' -modules

$$\det(\mathrm{Lie}(h))^\vee : \Omega_{G'} \rightarrow \Omega_G \otimes_R R'$$

that induces an isomorphism

$$\Omega_{G'} \otimes_{R'} K' \cong \Omega_G \otimes_R K'$$

by tensoring with K' . Thus we can view Ω_G as a sub- R -module of $\Omega_{G'} \otimes_{R'} K'$. This yields the following alternative description of the base change conductor.

3.3.1. Proposition. — *Let ϖ' be a uniformizer in R' . The base change conductor $c(G)$ of G is the unique element r of $\mathbb{Z}[1/d]$ such that*

$$(\varpi')^{rd} \omega_G$$

generates the R' -module $\Omega_{G'}$.

Proof. — Denote by \mathcal{G}' the Néron *lft*-model of G' . By definition, the length of the cokernel of the natural morphism

$$\mathrm{Lie}(h) : \mathrm{Lie}(\mathcal{G} \times_R R') \rightarrow \mathrm{Lie}(\mathcal{G}')$$

from (2.3.1) is equal to $c(G)d$. Writing $\mathrm{Lie}(h)$ in Smith normal form, it is easily seen that the cokernel of

$$\det(\mathrm{Lie}(h))^\vee : \Omega_{G'} \rightarrow \Omega_G \otimes_R R'$$

is isomorphic to $R'/(\varpi')^{c(G)d}$. □

3.3.2. Proposition. — *Let $R \rightarrow S$ be a flat local homomorphism of discrete valuation rings of ramification index one (in the sense of [4, 3.6.1]) and denote by L the quotient field of S . We denote by \mathcal{G}^L the Néron *lft*-model of $G \times_K L$.*

1. *The natural morphism*

$$\mathcal{G} \times_R S \rightarrow \mathcal{G}^L$$

is an isomorphism. In particular, it induces an isomorphism of S -modules

$$\Omega_G \otimes_R S \cong \Omega_{G \times_K L}.$$

2. *We have $c(G \times_K L) = c(G)$.*

This applies in particular to the case $S = \widehat{R^{sh}}$.

Proof. — (1) The formation of Néron *lft*-models commutes with the base change $R \rightarrow S$, by [4, 3.6.1].

(2) This follows easily from (1). \square

The semi-abelian K -variety G is bounded if and only if its Néron *lft*-model \mathcal{G} is of finite type over R [4, 10.2.1]. In that case, \mathcal{G} is called the Néron model of G . If G is bounded, then for every gauge form ω on G , we can consider the motivic integral

$$\int_G |\omega| \in \mathcal{M}_k^R.$$

In particular, we can consider the motivic integral of the “motivic Haar measure” $|\omega_G|$ associated to G . It does not depend on the choice of ω_G , since ω_G is unique up to multiplication with a unit in R .

3.3.3. Proposition. — *Let G be a bounded semi-abelian K -variety of dimension g with Néron model \mathcal{G} . Let ϖ be a uniformizer in R . Then for every integer γ , we have*

$$(3.3.1) \quad \int_G |\varpi^\gamma \omega_G| = \mathbb{L}^{-\gamma-g} [\mathcal{G}_k]$$

in \mathcal{M}_k^R . In particular, the virtual dimension of this motivic integral is equal to $-\gamma$.

Proof. — Since ω_G generates Ω_G , we have

$$\text{ord}_C(\varpi^\gamma \omega_G) = \gamma$$

for every connected component C of \mathcal{G}_k . Thus formula (3.2.1) becomes

$$\begin{aligned} \int_G |\varpi^\gamma \omega_G| &= \mathbb{L}^{-\gamma-g} \sum_{C \in \pi_0(\mathcal{G}_k)} [C] \\ &= \mathbb{L}^{-\gamma-g} [\mathcal{G}_k] \end{aligned}$$

in \mathcal{M}_k^R , where the last equality follows from the scissor relations in the Grothendieck ring. \square

3.4. Split subtori and bounded varieties. — We’ve already mentioned in Section 3.3 that a semi-abelian K -variety G is bounded if and only if the Néron *lft*-model \mathcal{G} of G is quasi-compact. If R is excellent (e.g., complete) and k algebraically closed, then this is also equivalent to the property that G does not contain a split torus [4, 10.2.1]. Since the boundedness condition plays an important role in the definition of the motivic integral, we’ll now take a closer look at split subtori of semi-abelian varieties. The results in this section will allow us to establish an equivalence between Question 2.4.1 and a Fubini property of motivic integrals (Theorem 4.2.1).

Let F be any field. We denote by (SpT/F) the category of split F -tori and by (SAb/F) the category of semi-abelian F -varieties (the morphisms in these categories are morphisms of algebraic F -groups).

For every semi-abelian F -variety G , we denote by G_{sp} the maximal split subtorus of G [14, 3.6]. If T is a split F -torus, then every morphism of F -groups $T \rightarrow G$ factors through G_{sp} , by [14, 3.5]. Thus we can define a functor

$$(\cdot)_{\text{sp}} : (\text{SAb}/F) \rightarrow (\text{SpT}/F) : G \mapsto G_{\text{sp}}.$$

For every semi-abelian F -variety G , we put $G^{\text{b}} = G/G_{\text{sp}}$. Then $(G^{\text{b}})_{\text{sp}}$ is trivial, by the remark after [14, 3.6]. It follows that every morphism of semi-abelian F -varieties $f : G \rightarrow H$ induces a morphism of semi-abelian F -varieties

$$f^{\text{b}} : G^{\text{b}} \rightarrow H^{\text{b}},$$

so that we obtain a functor

$$(\cdot)^{\text{b}} : (\text{SAb}/F) \rightarrow (\text{SAb}/F) : G \mapsto G^{\text{b}}.$$

3.4.1. Lemma. — *Let F be a field, and let $f : G \rightarrow H$ be a smooth morphism of semi-abelian K -varieties. Then the morphism $f_{\text{sp}} : G_{\text{sp}} \rightarrow H_{\text{sp}}$ is surjective.*

Proof. — The identity component of $G \times_H H_{\text{sp}}$ is a smooth and connected closed subgroup of G , and thus a semi-abelian F -variety [15, 5.2]. The morphism

$$(G \times_H H_{\text{sp}})^o \rightarrow H_{\text{sp}}$$

is still smooth. Therefore, we may assume that H is a split torus. It follows from [1, VI_B.1.2] that the image of f is closed in H , and it is also open by flatness of f . Thus f is surjective.

We denote by I the schematic image of the morphism $g : G_{\text{sp}} \rightarrow H$. This is a closed subgroup of the split torus H . The quotient H/I is again a split F -torus (it is geometrically connected because G_{sp} is geometrically connected, and it is smooth [1, VI_B.9.2(xii)] and diagonalizable [2, IX.8.1], so that it is a split torus). The quotient $Q = G/G_{\text{sp}}$ is an extension of an abelian F -variety and an anisotropic F -torus, so that the morphism of F -groups $Q \rightarrow H/I$ induced by f is trivial. But this morphism is surjective by surjectivity of f , so that H/I must be trivial, and $H = I$. Since the image of $G_{\text{sp}} \rightarrow H$ is closed [1, VI_B.1.2], it follows that $G_{\text{sp}} \rightarrow H$ is surjective. \square

3.4.2. Proposition. — *Let F be a field, and let*

$$0 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow 0$$

be an exact sequence of semi-abelian F -varieties.

1. *The schematic image of $(G_1)^{\text{b}} \rightarrow (G_2)^{\text{b}}$ is a semi-abelian subvariety H of G_2 , and the morphism $(G_1)^{\text{b}} \rightarrow H$ is an isogeny. Moreover, the sequence*

$$(3.4.1) \quad 0 \rightarrow H \rightarrow (G_2)^{\text{b}} \rightarrow (G_3)^{\text{b}} \rightarrow 0$$

is exact.

2. *If $(G_3)_{\text{sp}}$ is trivial, then*

$$(3.4.2) \quad 0 \rightarrow (G_1)^{\text{b}} \rightarrow (G_2)^{\text{b}} \rightarrow (G_3)^{\text{b}} \rightarrow 0$$

is exact.

Proof. — Dividing G_1 and G_2 by $(G_1)_{\text{sp}}$, we may assume that $(G_1)_{\text{sp}}$ is trivial (here we use that $G^b = (G/T)^b$ for every semi-abelian F -variety G and every split subtorus T of G). Then $(G_1)^b = G_1$.

First, we prove (1). The kernel of the morphism $G_1 \rightarrow (G_2)^b$ is the closed subgroup $\tilde{G}_1 = G_1 \times_{G_2} (G_2)_{\text{sp}}$ of G_1 . It is also a closed subgroup of $(G_2)_{\text{sp}}$. By [1, VI_B.9.2(xii)], the quotient $H = G_1/\tilde{G}_1$ is smooth over F . Since $(G_2)_{\text{sp}}$ is a split F -torus, we know that \tilde{G}_1 is a diagonalizable F -group [2, IX.8.1]. Since $(G_1)_{\text{sp}}$ is trivial, the F -group \tilde{G}_1 must be finite, so that the projection $G_1 \rightarrow H$ is an isogeny. The morphism $G_1 \rightarrow G_2$ induces a morphism of F -groups $H \rightarrow (G_2)^b$ that is a closed immersion [1, VI_B.1.4.2]. It identifies H with the schematic image of $G_1 \rightarrow (G_2)^b$. It follows from [15, 5.2] that H is a semi-abelian F -variety because it is a connected smooth closed subgroup of the semi-abelian F -variety $(G_2)^b$.

It is clear that (3.4.1) is exact at the right, so that it remains to prove that this sequence is also exact in the middle. By the natural isomorphism

$$(G_2/G_1)/((G_2)_{\text{sp}}/\tilde{G}_1) \cong (G_2/(G_2)_{\text{sp}})/(G_1/\tilde{G}_1)$$

it is enough to show that $\tilde{G}_2 = (G_2)_{\text{sp}}/\tilde{G}_1$ is the maximal split subtorus of $G_3 = G_2/G_1$. But $(G_2)_{\text{sp}} \rightarrow (G_3)_{\text{sp}}$ is surjective by Lemma 3.4.1, and its kernel is precisely \tilde{G}_1 , so we see that $\tilde{G}_2 = (G_3)_{\text{sp}}$.

Now we prove (2). Assume that $(G_3)_{\text{sp}}$ is trivial. Then the closed immersion $(G_2)_{\text{sp}} \rightarrow G_2$ factors through G_1 , and since $(G_1)_{\text{sp}}$ is trivial, we find that $(G_2)_{\text{sp}}$ must be trivial. Thus $G_i = (G_i)^b$ for $i = 1, 2, 3$, and the result is obvious. \square

If $(G_3)_{\text{sp}}$ is not trivial, it can happen that the sequence

$$0 \rightarrow (G_1)^b \rightarrow (G_2)^b \rightarrow (G_3)^b \rightarrow 0$$

in Proposition 3.4.2 is not left exact, as is shown by the following example.

3.4.3. Example. — Let K be the field $\mathbb{C}((t))$ of complex Laurent series and put $K' = K((\sqrt{t}))$. The Galois group $\Gamma = \text{Gal}(K'/K)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and it is generated by the automorphism σ that maps \sqrt{t} to $-\sqrt{t}$. Let G_2 be the K -torus with splitting field K' and character module

$$X(G_2) = \mathbb{Z} \cdot e_1 \oplus \mathbb{Z} \cdot e_2$$

where σ permutes e_1 and e_2 .

Let G_1 be the maximal anisotropic subtorus of G_2 . Its character module is $X(G_1) = X(G_2)/X(G_2)^\Gamma$. We put $G_3 = G_2/G_1$. This is a split K -torus with character module $X(G_3) = X(G_2)^\Gamma = \mathbb{Z} \cdot (e_1 + e_2)$.

For every K -torus T that splits over K' , we can consider the trace map

$$\text{tr}_T : X(T) \rightarrow X(T)^\Gamma : x \mapsto x + \sigma \cdot x.$$

It follows from the duality between tori and their character modules that the maximal split subtorus of T has character module $X(T)/\ker(\text{tr}_T)$ and that T^b is

the K -torus with character module $\ker(\mathrm{tr}_T)$. In this way, we see that $(G_1)_{\mathrm{sp}}$ is trivial and that $(G_2)^{\mathrm{b}}$ is the K -torus with character module

$$\ker(\mathrm{tr}_{G_2}) = \mathbb{Z} \cdot (e_1 - e_2).$$

Thus, applying the functor $(\cdot)^{\mathrm{b}}$ to the exact sequence of K -tori

$$0 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow 0,$$

we obtain the sequence

$$(3.4.3) \quad 0 \rightarrow G_1 \rightarrow (G_2)^{\mathrm{b}} \rightarrow 0 \rightarrow 0$$

and the morphism of K -tori $G_1 \rightarrow (G_2)^{\mathrm{b}}$ corresponds to the morphism of character modules

$$\alpha : \mathbb{Z} \cdot (e_1 - e_2) \rightarrow X(G_2)/X(G_2)^{\Gamma}.$$

The morphism α is injective but not surjective; its cokernel is

$$X(G_2)/(\mathbb{Z} \cdot (e_1 - e_2) + \mathbb{Z} \cdot (e_1 + e_2)) \cong \mathbb{Z}/2\mathbb{Z}$$

with trivial Γ -action. Therefore, (3.4.3) is not exact. More precisely, the morphism

$$G_1 \rightarrow (G_2)^{\mathrm{b}}$$

is an isogeny with kernel $\mu_{2,K}$.

3.4.4. Proposition. — *Assume that R is excellent and that k is algebraically closed. For every semi-abelian K -variety G , the quotient G^{b} is a bounded semi-abelian K -variety.*

Proof. — Since $(G^{\mathrm{b}})_{\mathrm{sp}}$ is trivial, this follows immediately from [4, 10.2.1]. \square

4. Chai's conjecture and Fubini properties of motivic integrals

4.1. Chai's conjecture and Haar measures. — Let

$$0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0$$

be a short exact sequence of semi-abelian K -varieties (as the notation suggests, the main example we have in mind is the Chevalley decomposition of a semi-abelian K -variety G as in Conjecture 1.1, but we will work in greater generality). The sequence of K -vector spaces

$$0 \rightarrow \mathrm{Lie}(T) \rightarrow \mathrm{Lie}(G) \rightarrow \mathrm{Lie}(A) \rightarrow 0$$

is exact, and by dualizing and taking determinants, we find a canonical isomorphism of K -vector spaces

$$\Omega_G \otimes_R K \cong (\Omega_T \otimes_R \Omega_A) \otimes_R K.$$

In this way, we can view $\Omega_T \otimes_R \Omega_A$ as an R -lattice in $\Omega_G \otimes_R K$.

The following proposition is implicit in the proof on pages 724–725 of [7] (proof of Proposition 4.1 in *loc. cit.* when the residue field is finite).

4.1.1. Proposition. — Assume that T is a torus. Let ω_T and ω_A be generators of Ω_T , resp. Ω_A . Let ϖ be a uniformizer of R , and denote by γ the unique integer such that $\varpi^{-\gamma}(\omega_T \otimes \omega_A)$ generates the R -module Ω_G . Then

$$c(G) = c(T) + c(A) + \gamma.$$

In particular, $c(G) - c(T) - c(A)$ belongs to \mathbb{Z} .

Proof. — By Proposition 3.3.2, we may assume that R is complete and that k is algebraically closed. Suppose that

$$\omega_G := \varpi^{-\gamma}(\omega_T \otimes \omega_A)$$

generates Ω_G . Let K' be a finite separable extension of K such that $G' = G \times_K K'$ has semi-abelian reduction, and denote by R' the normalization of R in K' . Then $A' = A \times_K K'$ has semi-abelian reduction and $T' = T \times_K K'$ is split [14, 4.1].

We denote by ϖ' a uniformizer of R' , and by d the ramification degree of the extension K'/K . By Proposition 3.3.1, the R' -module $\Omega_{G'}$ is generated by

$$(\varpi')^{c(G)d} \omega_G,$$

and the analogous property holds for A and T . We denote by \mathcal{G}' , \mathcal{T}' and \mathcal{A}' the Néron *lft*-models of G' , T' and A' , respectively. By the universal property of the Néron *lft*-model, the exact sequence

$$0 \rightarrow T' \rightarrow G' \rightarrow A' \rightarrow 0$$

extends uniquely to a sequence of R' -group schemes

$$0 \rightarrow \mathcal{T}' \rightarrow \mathcal{G}' \rightarrow \mathcal{A}' \rightarrow 0$$

and this sequence is exact by [7, 4.8(a)]. It follows that

$$\Omega_{G'} = \Omega_{T'} \otimes_{R'} \Omega_{A'} \subset \Omega_{G'} \otimes_{R'} K'$$

so that both $(\varpi')^{c(G)d} \omega_G$ and

$$(\varpi')^{(c(T)+c(A))d} (\omega_T \otimes \omega_A) = (\varpi')^{(c(T)+c(A))d} \varpi^\gamma \omega_G$$

are generators of the free R' -module $\Omega_{G'}$. Thus, we find that

$$(\varpi')^{(c(G)-c(T)-c(A))d} \varpi^{-\gamma}$$

is a unit in R' . This means that its ϖ' -adic valuation is zero, so that

$$c(G) = c(T) + c(A) + \gamma,$$

which concludes the proof. □

4.1.2. Remark. — Let

$$0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0$$

be an exact sequence of semi-abelian K -varieties, and let ω_T and ω_A be generators of Ω_T and Ω_A , respectively. In [7, §8.1], Chai considers the following statement:

(*) One has $c(G) = c(T) + c(A)$ if and only if $\omega_T \otimes \omega_A$ generates Ω_G .

If T is a torus, then this is a corollary of Proposition 4.1.1. However, if T is not a torus, it is not clear to us how statement $(*)$ can be proven, although Chai hints that it may be implicit in the proof on pages 724–725 of [7]. If G , T and A have semi-abelian reduction, then $c(G) = c(T) = c(A) = 0$ so that statement $(*)$ contains the following special case:

()** *If G , T and A have semi-abelian reduction, then $\omega_T \otimes \omega_A$ generates Ω_G .*

Even this property does not seem obvious, because the sequence of identity components of Néron *lft*-models

$$0 \rightarrow \mathcal{T}^o \rightarrow \mathcal{G}^o \rightarrow \mathcal{A}^o \rightarrow 0$$

might not be exact; see [4, 7.5.8] for an example where T , G and A are abelian varieties with good reduction and $\mathcal{T}^o \rightarrow \mathcal{G}^o$ is not a monomorphism. If statement **(**)** is true, then the proof of Proposition 4.1.1 shows that Proposition 4.1.1, and thus statement $(*)$, are valid without the assumption that T is a torus.

4.1.3. Lemma. — *Let G be a semi-abelian K -variety, and let T be a split subtorus of G . Then $c(G) = c(G/T)$. In particular, $c(G) = c(G^b)$.*

Proof. — We set $H = G/T$. By [7, 4.8(a)], the canonical sequence of group schemes

$$0 \rightarrow \mathcal{T} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$$

is exact, so that $\Omega_T \otimes \Omega_H = \Omega_G$. Now the result follows from Proposition 4.1.1 and the fact that $c(T) = 0$ because T has semi-abelian reduction. \square

4.2. Main results. — We can now state our main results.

4.2.1. Theorem. — *Let*

$$0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0$$

be an exact sequence of semi-abelian K -varieties, with T a torus. We put

$$\begin{aligned} \tilde{G} &= (G \times_K \widehat{K^{sh}})^b \\ \tilde{A} &= (A \times_K \widehat{K^{sh}})^b. \end{aligned}$$

and we denote by \tilde{T} the schematic image of the morphism

$$(T \times_K \widehat{K^{sh}})^b \rightarrow \tilde{G}.$$

Then \tilde{T} is a $\widehat{K^{sh}}$ -subtorus of \tilde{G} ,

$$0 \rightarrow \tilde{T} \rightarrow \tilde{G} \rightarrow \tilde{A} \rightarrow 0$$

is an exact sequence of bounded semi-abelian $\widehat{K^{sh}}$ -varieties, and

$$c(G) = c(T) + c(A)$$

if and only if

$$(4.2.1) \quad \dim \int_{\tilde{G}} |\omega_{\tilde{T}} \otimes \omega_{\tilde{A}}| = \dim \int_{\tilde{T}} |\omega_{\tilde{T}}| + \dim \int_{\tilde{A}} |\omega_{\tilde{A}}| = 0.$$

Proof. — By Proposition 3.3.2, we may assume that K is complete and k algebraically closed, so that $\widehat{K^{sh}} = K$. By Proposition 3.4.2, we know that T^b and \tilde{T} are isogenous K -tori, so that $c(T^b) = c(\tilde{T})$ by [8, 11.3 and 12.1]. Thus, by Proposition 3.4.2 and Lemma 4.1.3, we may assume that $\tilde{T} = T$, $\tilde{G} = G$ and $\tilde{A} = A$. Then T , G and A are bounded, by Proposition 3.4.4. so that we can take motivic integrals of gauge forms on T , G and A .

Let ϖ be a uniformizer in R . It follows from Proposition 3.3.3 that

$$\dim \int_T |\omega_T| + \dim \int_A |\omega_A| = 0$$

and that

$$\dim \int_G |\omega_T \otimes \omega_A|$$

is equal to the unique integer γ such that

$$\varpi^\gamma (\omega_T \otimes \omega_A)$$

generates the R -module Ω_G . By Proposition 4.1.1, we know that $\gamma = 0$ if and only if $c(G) = c(T) + c(A)$. \square

4.2.2. Remark. — In Conjecture 1.1, T is a torus and A is an abelian variety. This implies that A_{sp} is trivial, so that $A = A^b$ and

$$0 \rightarrow T^b \rightarrow G^b \rightarrow A \rightarrow 0$$

is exact by Proposition 3.4.2. In this case, in the proof of Theorem 4.2.1, we do not need the fact that the base change conductor of a torus is invariant under isogeny.

4.2.3. Theorem. — *Assume that K is complete and of characteristic zero and that k is algebraically closed. Let*

$$0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0$$

be an exact sequence of bounded semi-abelian K -varieties, with T a torus. Then

$$\dim \int_G |\omega_T \otimes \omega_A| = \dim \int_T |\omega_T| + \dim \int_A |\omega_A| = 0.$$

We will prove Theorem 4.2.3 in Section 5.4, using the model-theoretic approach to motivic integration in [10] and a new result on dimensions of motivic integrals (Theorem 5.2.1). As a corollary, we obtain a new proof of the following theorem of Chai.

4.2.4. Theorem (Chai). — *Let*

$$0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0$$

be an exact sequence of semi-abelian K -varieties, with T a torus. Assume that K is of characteristic zero. Then

$$c(G) = c(T) + c(A).$$

Proof. — This follows at once from Theorem 4.2.1 and Theorem 4.2.3. \square

4.2.5. Remark. — Theorem 4.2.3 is not a direct corollary of the Fubini theorem in [10] and the above results. We need to combine the Fubini result [10, 12.5] with the new result in Theorem 5.2.1 and its corollary below, which compares dimensions of motivic parameter integrals under rather general conditions. By the lack of a definable section for the morphism $G \rightarrow A$ as in Theorem 4.2.3, the motivic integral of $|\omega_G|$ over G may not be equal to the product of the integrals of $|\omega_T|$ over T and of $|\omega_A|$ over A . By the corollary to Theorem 5.2.1 and by the change of variables in [10, 12.4], this product survives at the rough level of virtual dimensions, which is sufficient to prove Theorem 4.2.3.

5. A comparison result for the dimensions of motivic integrals

In this section we will work with a specific context falling under [10] and define the dimension of motivic constructible functions at each point. These functions play an important role in motivic integration and in general Fubini results of [10] and [9]. In order to control dimensions as desired for equation (4.2.1) in Theorem 4.2.1, we will compare the dimensions of the integrals of possibly different functions F and G , when we are given that F and G have the same dimension in every point. Theorem 5.2.1 provides such a comparison result with parameters, and its corollary gives a similar comparison result for integrals on an algebraic variety with a volume form.

5.1. Dimensions of motivic constructible functions. — We suppose in this section that R is a complete discrete valuation ring of characteristic zero with quotient field K and algebraically closed residue field k of characteristic $p \geq 0$. We fix a uniformizer ϖ in R . We will use some terminology and results from [10], with precise references. Let $\mathcal{L}_{\text{high}}(K)$ be the Denef-Pas language $\mathcal{L}_{\text{high}}$ as in [10, §2.3] enriched with coefficients from K , and where the angular component maps $\overline{\text{ac}}_n$ for $n > 0$ are given by $\overline{\text{ac}}_n(x) = x\varpi^{-\text{ord}x} \bmod (\varpi)^n$ for nonzero $x \in K$. Let \mathcal{T} be the $\mathcal{L}_{\text{high}}(K)$ -theory of K .

Since \mathcal{T} falls under the combined Examples 1 and 4 of [10, §3.1], we can use the theory of motivic integration of [10]. Also, since \mathcal{T} is a complete theory, any definable subassignment X is uniquely determined by the definable set $X(K)$, with notation from [10, §4.1]. We will sometimes say definable set instead of definable subassignment.

We first define how to take (virtual) dimensions of several objects appearing in [10]. Write R_n for the ring $R/(\varpi^n)$ and ϖ_n for the image of ϖ in R_n . Let \mathcal{L}_r be the multisorted language with sorts R_n for integers $n > 0$, on each R_n the ring language with coefficients from R_n , and with the natural projection maps $p_{n,m} : R_n \rightarrow R_m$ for $n \geq m$. It follows from the quantifier elimination results of [22] and [23] that any $\mathcal{L}_{\text{high}}(K)$ -definable set $X \subset \prod_{i=1}^s R_{n_i}$ is already \mathcal{L}_r -definable, see also [10, Theorem 3.10]. To each \mathcal{L}_r -definable set $X \subset \prod_{i=1}^s R_{n_i}$, we associate an $\mathcal{L}_{\text{ring}}(k)$ -definable set $\delta(X)$ as follows. If R has mixed characteristic, the projection $p_{n,1}$ induces a bijection from the set of p^n -th powers in R_n to k , by Hensel's Lemma, Newton's binomium,

and the hypotheses on K . Let us write P_{p^n} for the set of p^n -th powers in R_n . Then any x in R_n can be written uniquely as

$$\sum_{i=0}^{n-1} x_i \varpi_n^i,$$

with $x_i \in P_{p^n}$, yielding a bijection $R_n \rightarrow k^n : x \mapsto (p_{n,1}(x_i))_i$ which is, in fact, \mathcal{L}_r -definable. If R has equal characteristic zero, we choose a retraction $k \rightarrow R$ of the ring morphism $R \rightarrow k$. This choice determines an isomorphism $R \rightarrow k[[\varpi]]$, and we identify $R_n \cong k[\varpi]/(\varpi^n)$ with the k -vector space k^n by means of the basis $1, \varpi, \dots, \varpi^{n-1}$ of R_n . In both cases, we obtain a bijection $\prod_{i=1}^s R_{n_i} \rightarrow k^N$ with $N = \sum_{i=1}^s n_i$. This identification maps the \mathcal{L}_r -definable subset X of $\prod_{i=1}^s R_{n_i}$ onto an $\mathcal{L}_{\text{ring}}(k)$ -definable subset of k^N that we denote by $\delta(X)$.

Recall from [10, §7.1] that $\mathcal{C}_+(\text{Point})$ is the Grothendieck semi-ring of $\mathcal{L}_{\text{high}}(K)$ -definable subsets of Cartesian products of the form $\prod_{i=1}^s R_{n_i}$ up to definable isomorphisms, with scissor relations, with zero-element $[\emptyset]$, and localized with respect to \mathbb{L} and the elements $\mathbb{L}^i - 1$ for all $i > 0$, where \mathbb{L} stands for the class of the affine line over k . Clearly, δ induces a semi-ring morphism

$$\mathcal{C}_+(\text{Point}) \rightarrow \mathcal{M}_k^+,$$

which we also denote by δ . Recall that objects in \mathcal{M}_k^+ have a dimension by Definition 3.1.1.

For an $\mathcal{L}_{\text{high}}(K)$ -definable set Z , $\mathcal{C}_+(Z)$ is a relative variant of $\mathcal{C}_+(\text{Point})$ over Z , see [10, §7.1]. An object $\varphi \in \mathcal{C}_+(Z)$ is called a motivic constructible function on Z . Moreover, for every $z \in Z(K)$, there is the evaluation map $i_z^* : \mathcal{C}_+(Z) \rightarrow \mathcal{C}_+(\text{Point})$ at z , and $i_z^*(\varphi)$ is called the evaluation of φ at z . For an $\mathcal{L}_{\text{high}}(K)$ -definable set Z , a point $z \in Z(K)$, and a function φ in $\mathcal{C}_+(Z)$, the dimension of φ at z is defined as $\dim(\delta(i_z^*(\varphi)))$ and is denoted by $\dim_z(\varphi)$. If Z is the point and $\varphi \in \mathcal{C}_+(\text{Point})$, we write $\dim(\varphi)$ instead of $\dim_{\text{Point}}(\varphi)$.

5.2. A comparison result. — In this section definable will mean for the language $\mathcal{L}_{\text{high}}(K)$. Recall that, for definable sets X, Y and $Z \subset X \times Y$, under integrability conditions in the fibers of the projection $Z \rightarrow X$, called X -integrability, one can integrate $\varphi \in \mathcal{C}_+(Z)$ in the fibers of the projection $Z \rightarrow X$ to obtain a function $\mu_X(\varphi)$ in $\mathcal{C}_+(X)$, see [10, 9.1]. The method of [9] and [10] for calculating integrals goes back to ideas by Denef in [11] in the p -adic case and to Pas [22], [23] in a pre-motivic setting.

Now we can state and prove our comparison result, stating that the dimension of a motivic integral only depends on the dimensions of the values of the integrand at each point.

5.2.1. Theorem. — *Let F and G be in $\mathcal{C}_+(Z)$ and suppose that $Z \subset X \times Y$ for some definable sets X, Y and Z . Suppose that F and G are X -integrable and that*

$$\dim_z(F) = \dim_z(G) \text{ for each point } z \text{ on } Z(K).$$

Then one has

$$\dim_x(\mu_{/X}(F)) = \dim_x(\mu_{/X}(G)) \text{ for each point } x \text{ on } X(K).$$

Proof. — By projecting one variable at the time and by iterating the one variable result, it suffices to consider the case where two of the three values n, m , and r , are zero, and either $n = 1, r = 1$, or $m = 1 = (1)$. By the cell decomposition theorem of [22], [23], we may suppose that $n = 0$. Indeed, via cell decomposition, each integral over a valued field variable is precisely calculated as a sum over \mathbb{Z} -variables and a subsequent integral over residue ring variables, see [10, §8].

Recall that F is a finite sum of terms of the form $a_i \otimes b_i$, with $a_i \in \mathcal{P}_+(Z)$ and $b_i \in \mathcal{Q}_+(Z)$, and similarly for G , with notation from [10, §7.1]. (The semi-ring $\mathcal{P}_+(Z)$ is related to the value group, $\mathcal{Q}_+(Z)$ to the residue field, and $\mathcal{C}_+(Z)$ is a tensor product of both.)

If $n = r = 0$, then we may suppose that the a_i lie in $\mathcal{P}_+(X)$, and similarly for G , by Proposition 7.5 of [10]. The result of the theorem now follows from the definition in [10, §6] of $\mu_{/X}$ in this case and the following simple comparison property for dimensions of constructible sets $A_i \subset \mathbb{A}_k^{n_i}$. If, for certain morphisms $f_i : \mathbb{A}_k^{n_i} \rightarrow \mathbb{A}_k^{n_3}$ for $i = 1, 2$, one has that $f_1(A_1) = f_2(A_2)$ and for each $x \in \mathbb{A}_k^{n_3}(k)$, the dimension of $f_1^{-1}(x) \cap A_1$ equals the dimension of $f_2^{-1}(x) \cap A_2$, then one has $\dim(A_1) = \dim(A_2)$.

Let us finally consider the case that $n = m = 0$ and $r = 1$. In this case we may suppose that the b_i lie in $\mathcal{Q}_+(X)$, and similarly for G , again by Proposition 7.5 of [10]. In the considered case, the theorem follows from the definition of $\mu_{/X}$ of [10] and the following two observations. For any $a \in \mathbb{A}$, where $a = a(\mathbb{L})$ is thus a rational function in \mathbb{L} of a specific kind, one has that $\dim(a)$ equals the degree of the rational function $a(\mathbb{L})$, where the degree of a rational function is the degree of its numerator minus the degree of its denominator, and where the degree of 0 is defined as $-\infty$. Secondly, there is the following elementary comparison property for the degrees of rational functions. Consider, for each $i \in \mathbb{Z}$, an integer $n_i \geq 0$, and a polynomial $a_i(x)$ over \mathbb{Z} in one variable x such that $a_i(q) \geq 0$ for each real $q > 1$. If there is a rational function $r(x)$ such that $\sum_{i \in \mathbb{Z}} a_i(q)/q^{n_i}$ converges and equals $r(q)$ for each real $q > 1$, then the following equality holds

$$\deg(r(x)) = \max_i \deg\left(\frac{a_i(x)}{x^{n_i}}\right),$$

where \deg stands for the degree. Since summation of non-negative functions over \mathbb{Z} in [10, §5] is calculated and defined by considering specific sums of rational functions in \mathbb{L} and by evaluating in real numbers $q > 1$, the result follows. \square

By working with affine charts over K one may consider a variety V over K as a definable subassignments, and one defines $\mathcal{C}_+(V)$ correspondingly, see [10, §12.3]. Let us write \int^{CL} for the integral as defined in [10, §12.3], to distinguish with the integrals from section 3.2 of this paper. The definition of these integrals in [10, §12.3] is based on finite affine covers of V over K , on finite additivity for motivic integrals, and on the change of variables formula.

5.2.2. Corollary. — *Let V be an algebraic variety over K with a volume form ω_V . Let F and G be integrable functions in $\mathcal{C}_+(V)$ such that, for each $x \in V(K)$, one has $\dim_x(F) = \dim_x(G)$. Then their integrals have the same dimension:*

$$\dim\left(\int_V^{\text{CL}} F|\omega_V|\right) = \dim\left(\int_V^{\text{CL}} G|\omega_V|\right).$$

Proof. — This follows immediately from Theorem 5.2.1 and the definition of the integrals \int^{CL} in [10, §12.3]. \square

5.3. Gelfand-Leray residues. — Let $f : X \rightarrow Y$ be a smooth morphism between smooth equidimensional varieties over K . Let m be the dimension of Y and let $m+n$ be the dimension of X . Let ω_X and ω_Y be differential forms of maximal degree on X and Y , respectively. Assume that ω_Y is a gauge form, that is, a generator of the line bundle $\Omega_{Y/K}^m$ at each point of Y .

Since f is smooth, the fundamental sequence of locally free coherent \mathcal{O}_X -modules

$$0 \rightarrow f^*\Omega_{Y/K}^1 \rightarrow \Omega_{X/K}^1 \rightarrow \Omega_{X/Y}^1 \rightarrow 0$$

is exact [13, 17.2.3]. Taking maximal exterior powers, we obtain an isomorphism

$$f^*\Omega_{Y/K}^m \otimes \Omega_{X/Y}^n \rightarrow \Omega_{X/K}^{m+n}.$$

Locally, this isomorphism is defined by

$$\phi \otimes \eta \mapsto \phi \wedge \tilde{\eta}$$

where $\tilde{\eta}$ is any lift of η to $\Omega_{X/K}^n$. Since $f^*\omega_Y$ generates the line bundle $f^*\Omega_{Y/K}^m$, we obtain an isomorphism

$$\Omega_{X/Y}^n \rightarrow \Omega_{X/K}^{m+n}$$

that is locally defined by

$$\eta \mapsto f^*\omega_Y \wedge \tilde{\eta}.$$

The inverse image of ω_X under this isomorphism is called the Gelfand-Leray form associated to ω_X and ω_Y , and denoted by ω_X/ω_Y . It induces a differential form of maximal degree on each of the fibers of f .

5.4. Proof of Theorem 4.2.3. — Finally, we can prove Theorem 4.2.3. It follows from Proposition 3.3.3 that

$$\dim \int_T |\omega_T| = \dim \int_A |\omega_A| = 0.$$

It is proven in Proposition 12.6 of [10] that the theory of motivic integration developed there can be used to compute the motivic integrals defined by the formula (3.2.1). In particular, the dimensions of the respective motivic integrals are the same, so that we can use the corollary of Theorem 5.2.1 and the results in [10] to prove Theorem 4.2.3.

We denote the projection morphism $G \rightarrow A$ by f . Since $H^1(K, T) = 0$ by [7, 4.3], the map $f(K) : G(K) \rightarrow A(K)$ is surjective. For every $a \in A(K)$, we set $G_a = f^{-1}(a)$. If we choose a point x in $G_a(K)$, then the multiplication by x defines

an isomorphism $\tau_x : T \rightarrow G_a$. Since the relative differential form $(\omega_T \otimes \omega_A)/\omega_A$ on G is invariant under translation, the pullback through τ_x of its restriction to G_a equals ω_T . Thus, by the change of variables formula in [10, 12.4], we have

$$\int_{G_a}^{\text{CL}} |(\omega_T \otimes \omega_A)/\omega_A| = \int_T^{\text{CL}} |\omega_T|$$

for each a in $A(K)$. Hence, by the Fubini property in [10, 12.5], we find that

$$\int_G^{\text{CL}} |\omega_T \otimes \omega_A| = \int_A^{\text{CL}} \psi |\omega_A|$$

where ψ is a motivic constructible function on A such that $\dim_a(\psi) = 0$ for each $a \in A(K)$. Now Theorem 5.2.2 with $V = A$ implies that

$$\dim \int_A^{\text{CL}} \psi |\omega_A| = \dim \int_A^{\text{CL}} |\omega_A| = 0.$$

Combining the above equations, we find

$$\dim \int_G |\omega_T \otimes \omega_A| = \dim \int_G^{\text{CL}} |\omega_T \otimes \omega_A| = 0,$$

which concludes the proof of Theorem 4.2.3.

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RAF CLUCKERS, Université Lille 1, Laboratoire Painlevé, CNRS - UMR 8524, Cité Scientifique, 59655 Villeneuve d'Ascq Cedex, France, and Katholieke Universiteit Leuven, Department of Mathematics, Celestijnenlaan 200B, B-3001 Leuven, Belgium

E-mail : Raf.Cluckers@math.univ-lille1.fr

Url : <http://math.univ-lille1.fr/~cluckers>

FRANÇOIS LOESER, Institut de Mathématiques de Jussieu, UMR 7586 du CNRS, Université Pierre et Marie Curie, Paris, France • *E-mail* : Loeser@math.jussieu.fr

Url : <http://www.math.jussieu.fr/~loeser/>

JOHANNES NICAISE, Katholieke Universiteit Leuven, Department of Mathematics, Celestijnenlaan
200B, B-3001 Leuven, Belgium • *E-mail* : `Johannes.Nicaise@wis.kuleuven.be`
Url : `http://wis.kuleuven.be/algebra/nicaise/`